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Triangulating multitolerance graphs

Andreas Parra *

Technische Universität Berlin, Fachbereich Mathematik, MA 6-1, Strasse des 17. Juni 136, 10623 Berlin, Germany

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Abstract

In this paper we introduce a new class of graphs which generalize both the tolerance and the trapezoid graphs, the *multitolerance graphs*. We show that the difference between the pathwidth and the treewidth of a multitolerance graph is at most one, and we develop an algorithm which solves the minimum fill-in problem for a multitolerance graph with a given representation in polynomial time. These results complement the recent results of Habib and Möhring [18, 25] about the treewidth and pathwidth of cocomparability graphs and graphs without asteroidal triples, and those of Kloks et al. [21] about the minimum fill-in problem. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Tolerance graphs were introduced in [15] by Golumbic and Monma as a generalization of both the interval and the permutation graphs. Recall that a graph $G=(V,E)$ is called a *tolerance graph* if there exists a family $\mathscr{I}=\{I_v=[l_v,r_v]: v\in V\}$ of closed intervals on the real line and a family $\mathscr{T}=\{t_v: v\in V\}$ of positive real numbers (the *tolerances*) satisfying

$$vw\in E \iff |I_v\cap I_w|\geqslant \min\{t_v,t_w\},$$

for all $v,w\in V$ with $v\neq w$, where $|I|$ denotes the length of the interval I. A tolerance graph is a *bounded tolerance graph* if in addition $t_v\leqslant |I_v|$ for all $v\in V$.

Golumbic et al. showed in [16] that bounded tolerance graphs are cocomparability graphs and that tolerance graphs are perfect. Moreover, they gave examples of tolerance graphs that are not cocomparability graphs. In the same paper the authors made the

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conjecture that a tolerance graph is bounded iff it is a cocomparability graph. This is still an open problem, but the conjecture is true for complements of trees [1, 12] (both papers contain a characterization of trees whose complements are tolerance graphs) and even for complements of bipartite graphs [26].

Bounded tolerance graphs are geometrically interpreted in [9] as *parallelogram graphs*, i.e., intersection graphs of parallelograms each of which has its horizontal lines on two parallel lines. Felsner [12] showed that parallelogram graphs and even all tolerance graphs that are cocomparability graphs, are a proper subclass of the *trapezoid graphs*, i.e., intersection graphs of trapezoids instead of parallelograms. These graphs are exactly the incomparability graphs of partial orders with interval dimension at most two [17]. While the recognition of tolerance and bounded tolerance graphs is still an open problem, trapezoid graphs can be recognized in polynomial time [22].

The *treewidth* (*pathwidth*) of a graph G is known to correspond to the smallest maximum clique size of all chordal graphs (interval graphs) that contain G as a partial subgraph, minus one. There are many applications mainly in computer science of these two notions [6, 24]. One important reason for this is that many $\mathcal{NP}$-complete graph problems are polynomially solvable for graphs with bounded tree- or pathwidth [2–4, 30].

In Section 3 we will introduce the *multitolerance graphs*, which generalize both the tolerance and the trapezoid graphs. (A related class of graphs appeared in [5] as the class of cocomparability graphs of bitolerance orders, which are identical to the trapezoid graphs.) Roughly speaking, multitolerance graphs allow two different tolerances for each vertex: one per each side of its interval.

In Section 4 we will prove by means of triangulating a graph (i.e., making a graph chordal by addition of some edges) that the pathwidth and treewidth of a multitolerance graph differ at most by one. This is supplementary to the main result of Habib and Möhring in [18], which states that the pathwidth and the treewidth of a cocomparability graph coincide. Recently it was shown by Möhring that this is valid even for graphs without asteroidal triples [25], which is a superclass of the class of cocomparability graphs.

The *Minimum Fill-In* problem 'Given a graph G, find a chordal graph that contains G s.t. the number of additional edges is minimum' plays a role in matrix factorization and has been investigated in [7, 29, 32]. It is known to be $\mathcal{NP}$-hard on cobipartite graphs [33] and polynomially solvable for bipartite permutation graphs [31] and cographs [10]. In Section 5 we generalize these results and present a polynomial time algorithm which solves Minimum Fill-In for multitolerance graphs. (Observe that this result would be covered by [21]. But, as the authors recently admitted, their algorithm does not work correctly.) To achieve this we will make use of the techniques of [18] and show the following: If all chordless cycles of a multitolerance graph are destroyed by addition of an inclusion-minimal set of chords, then no new chordless cycles are created, and thus arises a chordal graph.

It is known by [25] that, for graphs without asteroidal triples, Minimum Fill-In is the same as *Interval Completion*, i.e., Minimum Fill-In s.t. the chordal graph containing G

is an interval graph. Hence, our algorithm solves also the latter problem for trapezoid graphs.

2. Preliminaries

For graph and order-theoretic notions and properties of graph classes not given here we refer to [14, 23]. We consider only graphs that are simple, finite and undirected. The complement graph of $G=(V,E)$ is denoted by $\overline{G}=(V,\overline{E})$. For a subset $W \subseteq V$ we use $G[W]$ as a notation for the subgraph of G induced by W, and for $T \subseteq \overline{E}$ the graph $(V, E \cup T)$ is succinctly denoted by $G \cup T$. $Adj_G(v)$ is the set of vertices that are adjacent to v in G, and $N_G(v) = Adj_G(v) \cup \{v\}$.

Recall the definition of trapezoid graphs [11].

Defintion. A graph $G=(V,E)$ is called a *trapezoid graph* if there exist two families $\mathcal{I}^1 = \{[l_v^1, r_v^1] : v \in V\}$ and $\mathcal{I}^2 = \{[l_v^2, r_v^2] : v \in V\}$ of intervals on two parallel lines D^1 and D^2 satisfying

$$vw \in E \iff Q_v \cap Q_w \neq \emptyset,$$

for all $v, w \in V$ with $v \neq w$, where Q_x denotes the convex hull of $[l_x^1, r_x^1] \cup [l_x^2, r_x^2]$ in $\mathbb{R}^2$.

A *trapezoid order* associated with G is defined in [17] as the partial order $P = (V, <_P)$ obtained from the trapezoid representation of G by setting $v <_P w$ iff Q_v lies totally to the left of Q_w.

Next we come to the notions of treewidth and pathwidth as they were introduced by Robertson and Seymour [27, 28].

Definition. A *tree-decomposition* of a graph $G=(V,E)$ is a pair $(\{X_i : i \in I\}, T = (I,F))$ with $\{X_i : i \in I\}$ a family of subsets of V and T a tree, such that
(w1) $\bigcup_{i \in I} X_i = V$,
(w2) for all $e \in E$, there is some $i \in I$ s.t. $e \subseteq X_i$,
(w3) for all $v \in V$, $\{i \in I : v \in X_i\}$ induces a subtree in T.
The *width* of a tree-decomposition $(\{X_i : i \in I\}, T = (I,F))$ is $\max_{i \in I} |X_i| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all possible tree-decompositions of G. Such tree-decompositions with width $tw(G)$ are called *optimal.*

A *path-decomposition* of G is a tree-decomposition $(\{X_i : i \in I\}, T = (I,F))$ of G s.t. T is a path. The *pathwidth* of G, denoted by $pw(G)$, is defined analogously to the treewidth.

As a consequence, if $G \subseteq H$ (i.e., G is a partial subgraph of H), then $tw(G) \leqslant tw(H)$. The following property of tree-decompositions is well known, see e.g. [8].

Lemma 2.1 (Clique Containment Lemma). *Let* $(\{X_i : i \in I\}, T = (I,F))$ *be a tree-decomposition of* $G = (V,E)$ *and* $S \subseteq V$ *be a clique in* G. *Then there is some* $i \in I$ *with* $S \subseteq X_i$.

The following equivalent notions of treewidth and pathwidth can be found in [2, 20, 30], where $\omega(H)$ denotes the maximum size of a clique of H.

Lemma 2.2. *Let* G *be a graph. Then*
$tw(G) = \min\{\omega(H) : H \text{ chordal and } E(G) \subseteq E(H)\} - 1,$
$pw(G) = \min\{\omega(H) : H \text{ interval graph and } E(G) \subseteq E(H)\} - 1.$

This yields the next lemma.

Lemma 2.3. *Let* $H = (V,F)$ *be a chordal graph with* $G = (V,E) \subseteq H$ *and* $tw(G) = \omega(H) - 1$, *and let* $C \subseteq F \setminus E$. *Then* $tw(G) = tw(G \cup C) = tw(H)$.

For $G = (V,E)$ we define $\mathscr{C}(G) = \{C \subseteq \overline{E} : C$ is a $\subseteq$-minimal set of chords s.t. every induced C_4 of G has a chord in $G \cup C\}$. In [18] the following was shown implicitly.

Lemma 2.4. *If* G *is a cocomparability graph and* $C \in \mathscr{C}(G)$, *then* $G \cup C$ *is also a cocomparability graph.*

Finally, we remind the reader that the *minimum fill-in number* or *chordal completion number* $cc(G)$ of a graph G is the minimum number of additional edges that make G chordal, i.e.,

$$cc(G) = \min\{|E(H) - E(G)| : H \text{ chordal and } E(G) \subseteq E(H)\}.$$

A chordal graph that solves Minimum Fill-In for a graph G is not necessarily optimal for G w.r.t. treewidth, and v.v. Consider the graph G_8 as given in Fig. 1, and observe that G_8 contains 12 chordless cycles of length 4. Let $H = G_8 \cup \{a,b,c\}$ and $H' = G_8 \cup \{d,e,f,g\}$. Then H and H' are chordal graphs with $cc(G) = 3 = |E(H) - E(G)| < |E(H') - E(G)|$ and $tw(G) = 5 = \omega(H') - 1 < \omega(H) - 1$.

3. Multitolerance Graphs

Let $I = [l,r]$ be an interval on the real line and the numbers $lt, rt \in I$. We set

$$I_{lt,rt}(\lambda) = [l + (rt - l)\lambda, lt + (r - lt)\lambda] \text{ and } \mathscr{I}(I, lt, rt) = \{I_{lt,rt}(\lambda) : \lambda \in [0,1]\}.$$

So $\mathscr{I}(I, lt, rt)$ contains all intermediate intervals that we obtain when $[l, lt]$ is linearly transformed into $[rt, r]$. For an interval I, a set of *tolerance-intervals* τ is given by $\tau = \mathscr{I}(I, lt, rt)$ for some $lt, rt \in I$, or by the infinite tolerance-interval $\tau = \{\mathbb{R}\}$.

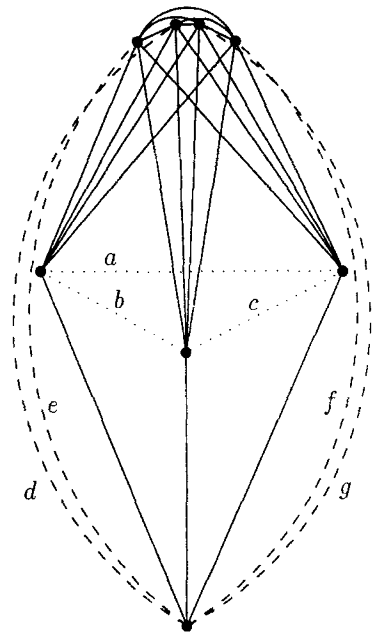

Fig. 1. The graph G_8, where $a, b, \ldots, g$ denote edges of $\overline{G_8}$.

Definition. A graph $G = (V, E)$ is called a *multitolerance graph* if there exists a family $\mathscr{I} = \{I_v = [l_v, r_v] : v \in V\}$ of intervals and a family $\mathscr{T} = \{\tau_v : v \in V\}$ of sets of tolerance-intervals satisfying

$$vw \in E \iff \begin{array}{l}\text{there is } T_v \in \tau_v \text{ with } T_v \subseteq I_w,\\ \text{or there is } T_w \in \tau_w \text{ with } T_w \subseteq I_v,\end{array}$$

for all $v, w \in V$ with $v \neq w$. A multitolerance graph is a *bounded multitolerance* graph if in addition $\tau_v \neq \{\mathbb{R}\}$ for all $v \in V$. Let $V_\infty = \{v \in V : \tau_v = \{\mathbb{R}\}\}$; hence V_∞ is an independent set. Let $V_0 = V \setminus V_\infty$, and for $v \in V_\infty$ define $W_v = \{w \in V : I_v \subseteq I_w \text{ and } vw \notin E\}$.

Observe that, unlike the definition of tolerance graphs, each vertex $v \in V$ either has two tolerance-intervals $L_v = [l_v, lt_v]$ and $R_v = [rt_v, r_v]$ on the left and right side of its interval $I_v = [l_v, r_v]$, or it has the infinite tolerance-interval $\mathbb{R}$, which corresponds to $t_v > |I_v|$ in the case of tolerance graphs. To allow the representation of an edge vw in the case that $w \in V_0, v \in V_\infty$ and $I_v \subseteq I_w$, we added all 'intermediate' intervals between L_w and R_w to τ_w, as defined above. See Fig. 2 as an example where $V_\infty = \{e\}$ and $W_e = \{c\}$.

If we would allow in the definition of multitolerance graphs that a vertex v with $I_v = [l_v, r_v]$ has a finite tolerance-interval only on one of the two sides of I_v, say e.g.

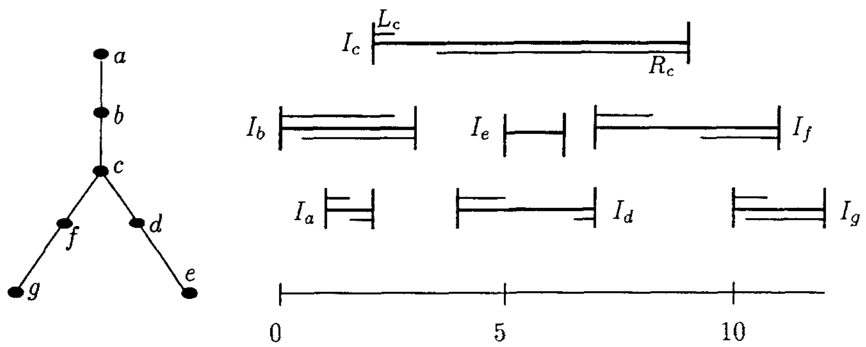

Fig. 2. The tree T_2 and its multitolerance representation.

$L_v = [l_v, lt_v]$ and $R_v = \{\mathbb{R}\}$, then we would not obtain a larger class of graphs. The represented graph would be the same as with the replacement of R_v by $[l_v, r_v]$.

In the following we assume for all multitolerance graphs $G = (V,E)$ that $\{l_v, lt_v, rt_v, r_v\} \cap \{l_w, lt_w, rt_w, r_w\} = \emptyset$ for all $v, w \in V$ with $v \neq w$. This can be achieved by small shifts of the involved points without any change in the graph that is represented. Refer to [16] for details in the case of tolerance graphs. Furthermore we assume that the set V_∞ is $\subseteq$-minimal, i.e., that for a vertex $v \in V_\infty$ the replacement of its tolerance-interval $\mathbb{R}$ by $[l_v, r_v]$, which is the longest possible bounded tolerance-interval, would create a new edge. Clearly, this is equivalent to $W_v \neq \emptyset$. Finally, we make the assumption that $Adj_G(v) \neq \emptyset$ for all $v \in V_\infty$, since otherwise v could be represented by an interval outside the range of $\mathscr{I}$.

Theorem 3.1. (a) *The trapezoid graphs are exactly the bounded multitolerance graphs.*

(b) *The class of tolerance graphs is properly contained in the class of the multi-tolerance graphs.*

(c) *Multitolerance graphs are perfect.*

Proof. (a) Let $(\mathscr{I}, \mathscr{T})$ be a representation of a bounded multitolerance graph $G = (V,E)$. Then $(\{[l_v, rt_v] : v \in V\}, \{[lt_v, r_v] : v \in V\})$ is a trapezoid representation of G, since two trapezoids intersect iff a tolerance-interval of one vertex is contained in the interval of the other one.

Conversely, let $(\mathscr{I}^1, \mathscr{I}^2)$ be a representation of a trapezoid graph $G = (V,E)$. We assume that the smallest point of all intervals of $\mathscr{I}^2$ is greater than the greatest one of all intervals of $\mathscr{I}^1$. This can be achieved without any change in the represented graph by shifting the intervals of $\mathscr{I}^1$ to the left or the intervals of $\mathscr{I}^2$ to the right. Then $(\{[l_v^1, r_v^2] : v \in V\}, \{\mathscr{I}([l_v^1, r_v^2], l_v^2, r_v^1) : v \in V\})$ is a bounded multitolerance representation of G.

(b) Let G be a tolerance graph with a representation $(\mathscr{I}, \mathscr{T})$. Then G is also a multitolerance graph with representation $(\mathscr{I}, \mathscr{T}' = \{\tau_v : v \in V\})$ where $\tau_v = \{\mathbb{R}\}$ if $t_v > |I_v|$, and $\tau_v = \mathscr{I}(I_v, l_v + t_v, r_v - t_v)$, otherwise, for all $v \in V$.

(c) The proof of [16] to show that tolerance graphs are perfect is also valid for multitolerance graphs. □

We state some simple properties of multitolerance graphs in the following lemma.

Lemma 3.2. *Let $G = (V,E)$ be a multitolerance graph.*
(a) *If $v \in V_\infty$ and $w \in W_v$, then $Adj_G(v) \subseteq Adj_G(w)$.*
(b) *G does not contain a chordless cycle of length greater than or equal to 5.*

Proof. (a) Let $x \in Adj_G(v)$. By definition there is $T_x \in \tau_x$ with $T_x \subseteq I_v \subset I_w$. Hence $x \in Adj_G(w)$.

(b) Obviously, the property of being a multitolerance graphs is hereditary. Thus it suffices to show that C_n, i.e., a chordless cycle on n vertices, is not a multitolerance graph for $n \geqslant 5$. As it is well-known that C_n is not a cocomparability graph, because of Theorem 3.1(a), it is not a bounded multitolerance graph. Part (a) of this lemma then implies that C_n cannot be a multitolerance graph that is not bounded. □

4. Bounds for Treewidth and Pathwidth

Lemma 4.1. *Let $G = (V,E)$ be a multitolerance graph and $C = \{xy \in \overline{E} : x,y \in Adj_G(v)$ for some $v \in V_\infty\}$. Then $G[V_0] \cup C$ is a cocomparability graph.*

Proof. Clearly, $G_0 = G[V_0]$ is a bounded multitolerance graph. Hence, because of Theorem 3.1(a) G_0 is a trapezoid graph with representation $(\{[l_v, rt_v] : v \in V\}, \{[lt_v, r_v] : v \in V\})$. Let $P = (V_0, <_P)$ be the trapezoid order associated with G_0, and let $R = (V_0, <_R)$ be the binary relation obtained from P by deleting all ordered pairs $x <_P y$ corresponding to the edges $xy \in C$.

To complete the proof we have to show that R is still a partial order. Suppose that this is not true. Since $<_R \subseteq <_P$ this only can happen if R is not transitive. So there are $x, y, z \in V$ with $x <_R y, y <_R z$ but $x \not<_R z$, and $x <_P z$ has been deleted since there is some $v \in V_\infty$ with $x, z \in Adj_G(v)$.

But then Q_y lies totally to the right of Q_x and totally to the left of Q_z, hence $l_v < rt_x$ and $r_v > lt_z$. This implies that $l_v < l_y$ and $r_v > r_y$, but $y \in Adj_G(v)$ contradicts $x <_R y$. □

A *caterpillar* is a tree which does not contain a subtree isomorphic to the tree T_2 in Fig. 2.

Theorem 4.2. *Let $G = (V,E)$ be a multitolerance graph. Then*

$$pw(G) \leqslant tw(G) + 1.$$

Moreover, there is an optimal tree-decomposition of G whose tree is a caterpillar.

Proof. Let $H = (V,F)$ be a chordal graph with $E \subseteq F$ and $tw(G) = \omega(H) - 1$. Then every induced C_4 of G has a chord in H. Let $C \subseteq F \setminus E$ be a $\subseteq$-minimal set of such chords; thus $C \in \mathscr{C}(G)$. Define $V_1 = \{v \in V_\infty : Adj_G(v) \text{ is a clique in } G \cup C\}$, $V_2 = V_\infty \setminus V_1$, and, for all $v \in V_\infty$,

$$C_v = \begin{cases} \{xy \in C : x, y \in Adj_G(v)\} & \text{if } v \in V_1, \\ \{vw \in C : w \in W_v\} & \text{if } v \in V_2. \end{cases}$$

Next we show the following claims:
(1) for all $v \in V_1$ and $w \in V$, $vw \notin C$.
(2) for all $v \in V_2$, $C_v = \{ vw \in \overline{E} : w \in W_v\}$.
For (1), suppose that there is $v \in C$ with $v \in V_1$. Hence, there is an induced C_4 (v,x,w,y) of G with $x, y \in Adj_G(v)$ and, because of the $\subseteq$-minimality of C, $xy \notin C$. This contradicts $v \in V_1$.

For (2), let $v \in V_2, w \in W_v$. Then there are $x, y \in Adj_G(v)$ s.t. $xy \notin E \cup C$, which, because of Lemma 3.2(a) implies that $vw \in C$, since every induced C_4 of G has a chord in H.

Let $C_1 = \bigcup_{v\in V_1} C_v$, $C_2 = \bigcup_{v\in V_2} C_v$. Because of (1) and (2) and the definition of W_v, $G_1 = G[V \setminus V_1] \cup C_2$ is a bounded multitolerance graph s.t. the infinite tolerance-interval $\mathbb{R}$ is replaced by I_v, for all $v \in V_2$. Thus $G_2 = G \cup C_2$ is a multitolerance graph with $V_\infty(G_2) = V_1$. Hence, we can apply Lemma 4.1 to G_2. Because of (1), this implies that $K = G_1 \cup C_1$ is a cocomparability graph. From Lemma 2.3 and from the facts that the pathwidth and the treewidth of a cocomparability graph coincide [18] and that K is an induced subgraph of $G \cup C_1 \cup C_2$, we conclude that

$$pw(K) = tw(K) \leqslant tw(G \cup C_1 \cup C_2) = tw(G).$$

Consider an optimal path-decomposition $(\{X_i : i \in I\}, P = (I,F))$ of K. The Clique Containment Lemma 2.1 implies that for every $v \in V_1$, there is an $i(v) \in I$ s.t. $Adj_G(v) \subseteq X_{i(v)}$. Now we add, successively for every $v \in V_1$, a new vertex j that corresponds to the set $(X_{i(v)} \cup \{v\})$ between $i(v)$ and one of its neighbours in P, and obtain a new path $P = (I,F)$. Finally, $(\{X_i : i \in I\}, P = (I,F))$ is a path-decomposition of $G \cup C_1 \cup C_2$ and therefore one of G, too. Its pathwith p is at most one more than the pathwith of K. As a result, $p \leqslant tw(G) + 1$.

To obtain an optimal tree-decomposition of G whose tree is a caterpillar, we only have to attach $X_{i(v)} \cup \{v\}$ to $X_{i(v)}$ in P, for every $v \in V_1$. Clearly, this is a tree-decomposition of G, and it is optimal, since P is optimal for K and because of Lemma 2.1. □

Remark. The upper bound $tw(G) + 1$ for the pathwidth of G is sharp since for the tree T_2 in Fig. 2 we have $pw(T_2) = 2$ and $tw(T_2) = 1$.

Corollary 4.3. *If $G = (V,E)$ is a multitolerance graph and $C \in \mathscr{C}(G)$, then $G \cup C$ does not contain an induced C_n, for any $n \geqslant 5$.*

Proof. Define V_1, V_2, C_1, C_2, and K as in the proof of Theorem 4.2. First we show:

(1) $C \setminus C_1 \in \mathscr{C}(G \cup C_1)$.

Clearly, by the $\subseteq$-minimality of C, there is an induced C_4 of $G \cup C_1$ that contains e, for every $e \in C \setminus C_1$, and that remains chordless in $G \cup C \setminus \{e\}$. So to prove (1) it is sufficient to show that no new induced C_4 arose by the addition of C_1 to G.

Suppose that there is an induced C_4 of $G \cup C_1$, say $R = (x, y, a, b)$, that has not been a C_4 in G. W.l.o.g., let $xy \in C_1$. Thus we can find $v \in V_1$ with $x, y \in Adj_G(v)$. Since $Adj_G(v)$ is a clique in $G \cup C_1$, $a, b \notin N_G(v)$. Now replace (x, y) in R by (x, v, y), and replace any other edge e of R with $e \in C_1$ analogously. Finally, because V_1 is an independent set of G, we obtain an induced cycle R of G with at least 5 vertices –a contradiction to Lemma 3.2(b). We now prove:

(2) $\mathscr{C}(G[V \setminus V_1] \cup C_1) = \mathscr{C}(G \cup C_1)$.

Since $Adj_G(v)$ is a clique in $G \cup C_1$, for every $v \in V_1$, there is no induced C_4 of $G \cup C_1$ that contains v. Thus V_1 can be removed. For the proof of (3) recall that $K = G[V \setminus V_1] \cup C_1 \cup C_2$.

(3) $C \setminus (C_1 \cup C_2) \in \mathscr{C}(K)$.

Suppose now that there is an induced C_4 (v, w, a, b) of K that remains chordless in $K \cup C \setminus (C_1 \cup C_2)$ and thus, because of (1) and (2), has not been a C_4 in $G[V \setminus V_1] \cup C_1$. W.l.o.g., let $vw \in C_2$ with $v \in V_2$ and $w \in W_v$. Lemma 3.2.a implies that $b \notin Adj_G(v)$, hence $vb \in C_2$ and $b \in W_v$. Observe that $vb \in C_1$ is not possible, as $v \notin Adj_G(v')$ for any $v' \in V_\infty$, and $v \in W_b$ with $b \in V_2$ can not hold, since this would imply that $w \in W_b$ and thus $bw \in C_2$.

Because of $v \in V_2$, there are $x, y \in Adj_G(v)$ with $xy \notin E \cup C$. Consequently, (x, w, y, b) is an induced C_4 of G that must have a chord in $G \cup C$, but $wb \in C$ is a contradiction.

Since K is a cocomparability graph, (3) and Lemma 2.4 imply that $K \cup C$ –which is the same as $G[V \setminus V_1] \cup C$– is also a cocomparability graph. It is well-known that a cocomparability graph does not contain an induced C_n, for any $n \geqslant 5$; this is valid for $G \cup C$, too, because $Adj_G(v)$ is a clique in $K \cup C$, for every $v \in V_1$. □

5. Minimum Fill-In

In this section we give a proof of the following theorem.

Theorem 5.1. *Given a multitolerance graph $G = (V, E)$ and a multitolerance representation of G, Minimum Fill-In for G can be solved in $\mathcal{O}(|V|^5)$ time.*

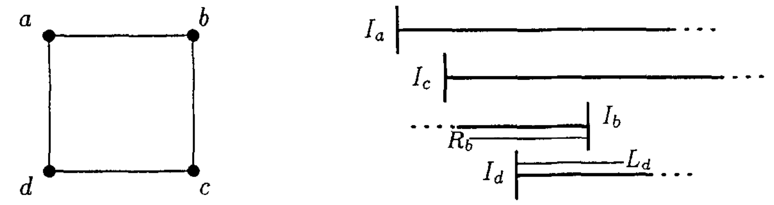

Fig. 3. A C_4 and its multitolerance representation.

First we investigate the possibilities of how to represent a C_4 (a,b,c,d) as a multitolerance graph. W.l.o.g., let $l_a < l_c$ and $l_b < l_d$. Furthermore, we assume $l_c < l_d$, and let $x \in \{a,c\}$ s.t. $r_x = \min\{r_a, r_c\}$, and $V = \{a,b,c,d\}$. See Fig. 3.

Proposition 5.2. (a) $(I_b \cap I_d) \subset (I_a \cap I_c)$.
(b) *$b,d \in V_0$ and $R_b, L_d \subset (I_a \cap I_c)$.*

Proof. (a) Clearly, $I_a \cap I_c \neq \emptyset$, since otherwise $I_a \cap I_d = \emptyset$. Suppose that $(I_b \cap I_d) \not\subset (I_a \cap I_c)$, i.e., $r_x < \min\{r_b, r_d\}$. Either $x \in V_\infty$ or $x = a$ and $rt_x < l_c < l_d$, since otherwise ac would be represented. Analogously, we have either $d \in V_\infty$ or $lt_d > r_b > r_x$. Hence, the edge xd is not represented, a contradiction.

(b) Because a is not adjacent to c, $I_a \cap I_c$ contains no tolerance-interval of a or c. Hence, it is not possible that $I_b \cap I_d = [l_d, r_d]$, because this would imply that $d \in V_\infty$ and that the edge ad is not represented. Thus $I_b \cap I_d = [l_d, r_b]$. Now the only way to represent the edge bc (resp. dx) is that $rt_b > l_c$ (resp. $lt_d < r_x$). □

Obviously, the assumption "$l_d < l_c$" would have led us to analogous results, where "b" and "d" are replaced by "a" and "c", and v.v.

For a multitolerance graph $G = (V,E)$ let $D = \{ac \in \overline{E} : \text{there are } b,d \in V \text{ s.t. } (a,b,c,d) \text{ is an induced } C_4 \text{ of } G\}$. We define a binary relation '$<$' on D that will be shown to be an order relation. Given a multitolerance representation of G, let for all $bd, ac \in D$:

$$bd < ac \iff (a,b,c,d) \text{ is an induced } C_4 \text{ of } G \text{ and } (I_b \cap I_d) \subset (I_a \cap I_c).$$

As a consequence of Proposition 5.2, for every induced C_4 (a,b,c,d) of G, either $bd < ac$ or $ac < bd$, and if $bd < ac$, then $b,d \in V_0$. So there is a bijection between the induced C_4s of G and the ordered pairs of $(D, <)$.

Lemma 5.3. *Let $G = (V,E)$ be a multitolerance graph. Then $(D, <)$ is a partial order.*

Proof. Clearly, $(D, <)$ is irreflexive, so we only have to show transitivity. Let $ef < bd$ and $bd < ac$. Because of Proposition 5.2, there are $T_e \in \tau_e, T_f \in \tau_f$ with $T_e, T_f \subset (I_b \cap I_d) \subset (I_a \cap I_c)$. Hence $\{e,f\} \neq \{a,c\}$, and $(a,e,c,f,)$ is an induced C_4 of G with $ef < ac$. □

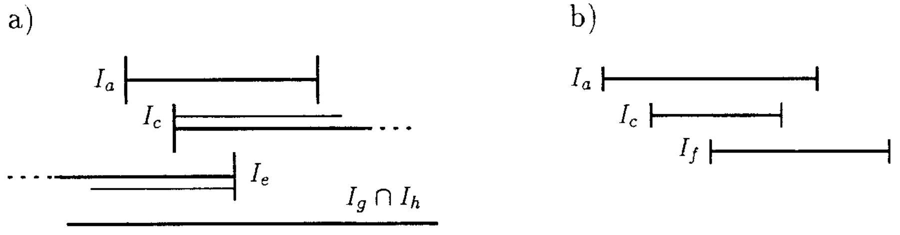

Fig. 4.

Observe that the statement of the following lemma is not true for arbitrary graphs, e.g. not for the cocomparability graph $\overline{C_6}$.

Lemma 5.4. *If $G = (V,E)$ is a multitolerance graph and $C \in \mathscr{C}(G)$, then $G \cup C$ is a chordal graph.*

Proof. Because of Lemma 3.2(b) and Corollary 4.3 it suffices to show that $H = G \cup C$ contains no induced C_4. Suppose that there is an induced C_4 (a,c,e,f) of H. Hence, one of its edges must be a chord of an induced C_4 of G. W.l.o.g., let $ac \in C$, $l_a < l_c$, and (a,b,c,d) be an induced C_4 of G that remains chordless in $G \cup (C \setminus \{ac\})$. Such a C_4 exists because of the $\subseteq$-minimality of C, and it will be called an ac-C_4.

Because of their adjacencies, $\{b,d\} \cap \{e,f\} = \emptyset$. Observe that ae, cf, and bd are not in $E \cup C$. Now we consider several multiply branched cases which we will each lead separately to a contradiction.

Case 1: $l_e < l_a$.

Case 1.1: $r_e > r_a$.

Then $I_a \subset I_e$ and $a \in V_\infty$, $e \in W_a$. From Lemma 3.2(a) we obtain $Adj_G(a) \subseteq Adj_G(e)$ and thus an induced C_4 (a,b,e,d) of G with $ae, bd \notin C$, a contradiction. In the following we will call such a C_4 of G that remains chordless in $G \cup C$, a *con-C_4* (abbreviation for contradictory C_4).

Case 1.2: $r_e < r_a$.

Then $I_a \supset (I_e \cap I_c)$. Hence, $ce \in E$ would imply that $ac \in E$ or $ae \in E$. Thus $ce \in C$, and there is a ce-C_4, namely (c,g,e,h).

- *Case* 1.2.1: $ce < gh$. See Fig. 4(a).

  Proposition 5.2 yields $e,c \in V_0$ and $R_e, L_c \subset (I_g \cap I_h)$. Because of $rt_e < l_a < r_a < lt_c$, we obtain $I_a \subset (I_g \cap I_h)$. In the case $a \in V_0$ we have a con-C_4 (a,g,e,h).

  On the other side, if $a \in V_\infty$, then Proposition 5.2 implies that $bd < ac$ and that there are $T_b \in \tau_b$ and $T_d \in \tau_d$ s.t. $T_b, T_d \subset I_a$. Hence $T_b, T_d \subset (I_g \cap I_h)$, and, as a result, (b,g,d,h) is a con-C_4.

- *Case* 1.2.2: $gh < ce$.

  Then Proposition 5.2 implies that $g,h \in Adj_G(a)$. Hence, (g,e,h,a) is a con-C_4.

Case 2: $r_f > r_c$.

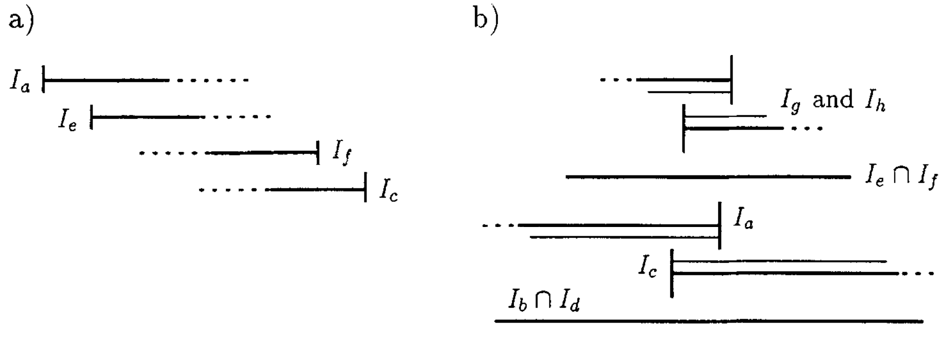

Fig. 5.

This case is analogous to case 1, apart from the additional possibility that $l_f > l_c$ and $I_c \subset I_a$; see Fig. 4(b).

Then $c \in V_\infty$, $bd < ac$, and, because of Lemma 3.2(a), $ce \notin E$. Thus there is a ce-C_4 (c,g,e,h) with $gh < ce$. Hence $g,h \in Adj_G(a)$, and (g,e,h,a) is a con-C_4.

Case 3: $l_e > l_a$ and $r_f < r_c$. See Fig. 5(a).

Case 3.1: $ef \in E$.

W.l.o.g., we assume that there is $T_e \in \tau_e$ with $T_e \subseteq I_f$. Hence, $e \in V_0$ and $r_e > r_a$.

- *Case* 3.1.1: $I_f \subset I_c$.

  Then $f \in V_\infty$ and $af \notin E$. Thus there is an af-C_4 (a,g,f,h) with $gh < af$, which yields $g,h \in Adj_G(c)$. Hence, (g,f,h,c) is a con-C_4.
- *Case* 3.1.2: $l_f < l_c$ and $ac < bd$.

  Then $a,c \in V_0$. Hence, $l_e > rt_a$ and $r_f < lt_c$. Consequently and because of Proposition 5.2 and since $l_a < l_c$, $T_e \subseteq [l_e, r_f] \subset [rt_a, lt_c] \subset (I_b \cap I_d)$, which yields a con-$C_4$ (b,a,d,e).
- *Case* 3.1.3: $l_f < l_c$ and $bd < ac$.

  Since $r_f < r_a$ would imply that $T_e \subset I_a$, we have $r_a < r_f$. Hence, $(I_a \cap I_c) \subset I_f$, and $b,d \in Adj_G(f)$. As a result, (b,c,d,f) is a con-C_4.

Case 3.2: $ef \notin E$.

Let (e,g,f,h) be an ef-C_4. Because of their adjacencies, $\{g,h\} \cap \{a,c\} = \emptyset$.

- *Case* 3.2.1: $gh < ef$. See Fig. 5(b).

  Clearly, $(I_e \cap I_f) \subset [l_a, r_c]$. $(I_e \cap I_f) \subset I_a$ (resp. I_c) would imply that $g,h \in Adj_G(a)$ (resp. $Adj_G(c)$), and, consequently, (g,e,h,a) (resp. (g,e,h,c)) would be a con-C_4.

  Hence $(I_e \cap I_f) \supset (I_a \cap I_c)$. If $bd < ac$, then $b,d \in Adj_G(e)$, and (b,a,d,e) would be a *con*-C_4. Thus $ac < bd$, which yields $rt_a, lt_c \notin (I_e \cap I_f)$. Hence $g,h \in (Adj_G(b) \cap Adj_G(d))$, and (g,b,h,d) is a con-C_4.
- *Case* 3.2.2: $ef < gh$ and $bd < ac$.

  Hence $e,f \in V_0$ and $lt_e > r_a$, $rt_f < l_c$. If $l_e < l_f$, then, because of $r_e > r_a$ and $l_f < l_c$, $(I_a \cap I_c) \subset I_e$. This would imply that $b,d \in Adj_G(e)$, and (b,a,d,e) would be a con-C_4.

Thus $l_f < l_e$. Because of Proposition 5.2, $(I_a \cap I_c) \subset [rt_f, lt_e] \subset (I_g \cap I_h)$. Consequently, $b, d \in (Adj_G(g) \cap Adj_G(h))$, and (b, g, d, h) is a con-C_4.

- *Case* 3.2.3: $ef < gh$ *and* $ac < bd$.

  Then $a, c, e, f \in V_0$ and $l_e > rt_a$, $rt_f < l_c$. If $lt_e < lt_c$, then, because of $[l_e, lt_e] \subset [rt_a, lt_c] \subset (I_b \cap I_d)$, (b, a, d, e) would be a con-C_4.

  Thus $lt_e > lt_c$. Because of $r_f < lt_c$ and $I_f \not\subset I_e$, this implies that $l_f < l_e$, and therefore $[l_c, lt_c] \subset [rt_f, lt_e] \subset (I_g \cap I_h)$, which yields a con-$C_4$ (g, f, h, c). □

Now we are ready to prove the theorem.

Proof of Theorem 5.1. The partial order $(D, <)$ can be obtained in $\mathcal{O}(|V|^4)$ time, since we have to check for all $ac, bd \in \overline{E}$ if $ac < bd$. First we show:

(1) $C \in \mathscr{C}(G) \iff D \setminus C$ is a $\subseteq$-maximal antichain of $(D, <)$.

Clearly, because there is a bijection between the induced C_4 of G and the ordered pairs of $(D, <)$, a set C of C_4-chords destroys all induced C_4s of G iff $D \setminus C$ is an antichain of $(D, <)$. The $\subseteq$-minimality of C corresponds exactly to the $\subseteq$-maximality of $D \setminus C$.

Lemma 5.4 states that $G \cup C$ is a chordal graph, for every $C \in \mathscr{C}(G)$. Since every induced C_4 of G has a chord in a chordal graph that contains G, this implies that

(2) $cc(G) = \min_{C \in \mathscr{C}(G)} |C|$.

Hence, Minimum Fill-In is equivalent to finding a maximum antichain of $(D, <)$. Since the latter problem is equivalent to maximum matching in a bipartite graph [13], this can be done in $\mathcal{O}(|D|^{5/2})$ time [19]. □

Remark. As mentioned in the Introduction, Minimum Fill-In and Interval Completion are the same in particular for trapezoid graphs. Hence, the algorithm to prove Theorem 5.1 solves also Interval Completion, if the input graph $G = (V, E)$ is a bounded multitolerance graph. Moreover, there is no need to demand a given representation in this case, since a trapezoid representation can be found in $\mathcal{O}(|V|^2)$ time [22].

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